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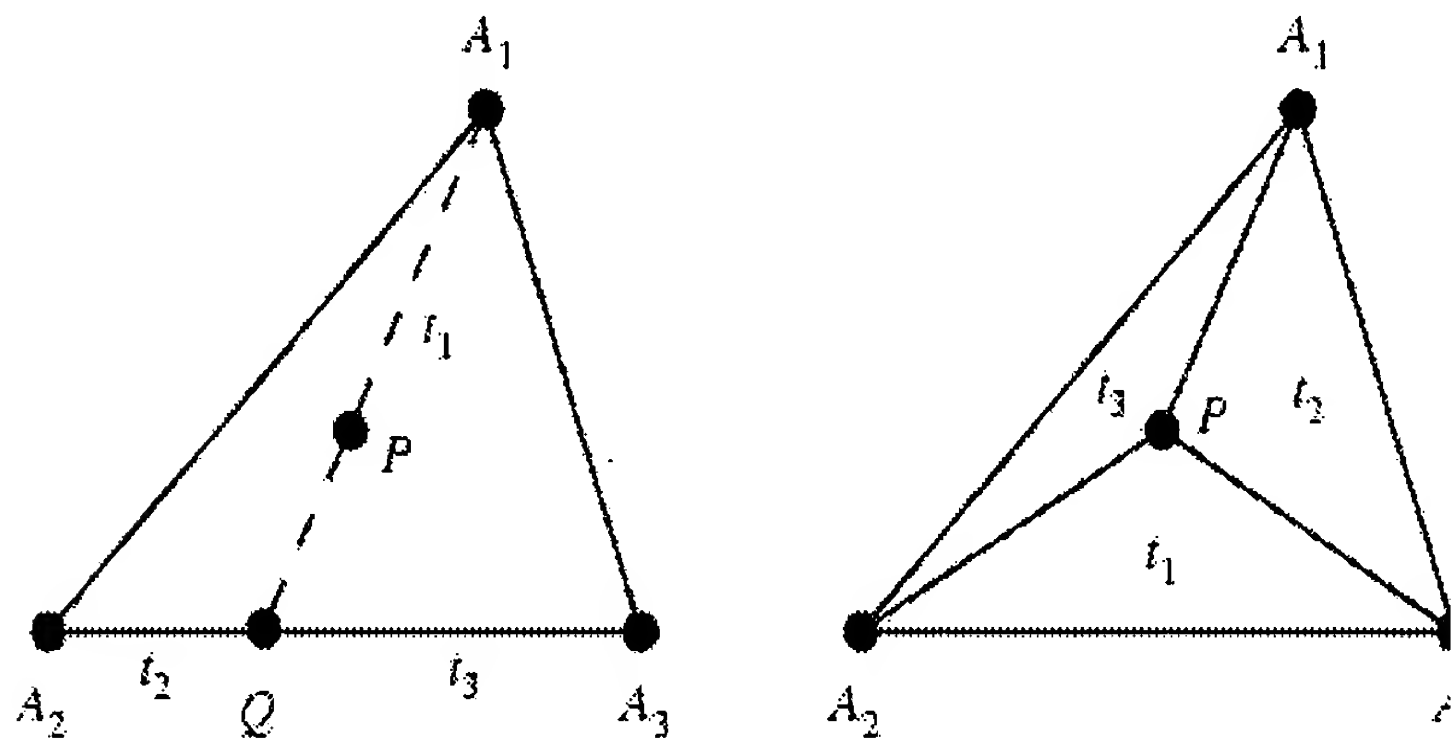
Barycentric Coordinates

Barycentric coordinates are triples of numbers (t_1, t_2, t_3) corresponding to masses placed at the vertices of a reference triangle $\Delta A_1 A_2 A_3$. These masse then determine a point P, which is the geometric centroid of the three masses, and is identified with coordinates (t_1, t_2, t_3). The vertices of the triangle are given by $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Barycentric coordinates were discovered by Möbius in 1827 (Coxeter 1969, p. 217; Fauvel *et al.* 1993).



To find the barycentric coordinates for an arbitrary point P, find t_2 and t_3 from the point Q at the intersection of the line $A_1 P$ with the side $A_2 A_3$, and then determine t_1 as the mass at A_1 that will balance a mass $t_2 + t_3$ at Q, thus making P the centroid (left figure). Furthermore, the areas of the triangle $\Delta A_1 A_2 P$, $\Delta A_1 A_3 P$, and $\Delta A_2 A_3 P$ are proportional to the barycentric coordinates t_3, t_2, and t_1 of P (right figure; Coxeter 1969, p. 217).

Barycentric coordinates are homogeneous, so

$$(t_1, t_2, t_3) = (\mu t_1, \mu t_2, \mu t_3) \qquad (1$$

for $\mu \neq 0$. Barycentric coordinates normalized so that they become the actua

areas of the subtriangles are called homogeneous barycentric coordinates, anc barycentric coordinates normalized so that

$$t_1 + t_2 + t_3 = 1, \qquad (2$$

so that the coordinates give the areas of the subtriangles *normalized by the area of the original triangle* are called areal coordinates (Coxeter 1969, p. 218). Barycentric and areal coordinates can provide particularly elegant proofs of geometric theorems such as Routh's theorem, Ceva's theorem, and Menelaus' theorem (Coxeter 1969, pp. 219-221).

(Not necessarily homogeneous) barycentric coordinates for a number of common centers are summarized in the following table.

triangle center	barycentric coordinates
circumcenter O	$(a^2(b^2+c^2-a^2), b^2(c^2+a^2-b^2), c^2(a^2+b^2-c^2))$
excenter J_A	$(-a, b, c)$
excenter J_B	$(a, -b, c)$
excenter J_C	$(a, b, -c)$
Gergonne point Ge	$((s-b)(s-c), (s-c)(s-a), (s-a)(s-b))$
incenter I	(a, b, c)
Nagel point Na	$(s-a, s-b, s-c)$
orthocenter H	$((a^2+b^2-c^2)(c^2+a^2-b^2),$ $(b^2+c^2-a^2)(a^2+b^2-c^2),$ $(c^2+a^2-b^2)(b^2+c^2-a^2))$
symmedian point K	(a^2, b^2, c^2)
triangle centroid G	$(1, 1, 1)$

In barycentric coordinates, a line has a linear homogeneous equation. In

particular, the line joining points (r_1, r_2, r_3) and (s_1, s_2, s_3) has equation

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{vmatrix} = 0 \qquad (3$$

(Loney 1962, pp. 39 and 57; Coxeter 1969, p. 219; Bottema 1982). If the vertices P_i of a triangle $\Delta P_1 P_2 P_3$ have barycentric coordinates (x_i, y_i, z_i), then the area of the triangle is

$$\Delta P_1 P_2 P_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta ABC \qquad (4$$

(Bottema 1982, Yiu 2000).

SEE ALSO: Areal Coordinates, Exact Trilinear Coordinates, Homogeneous Barycentric Coordinates, Trilinear Coordinates

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On-Line Computer Graphics Notes

Barycentric Coordinates

If we are given a frame in three-dimensional space we know how to define a local coordinate system with respect to the frame. However, given a set of points in three-dimensional space, we can also define a local coordinate system with respect to the these points. These coordinate systems are called *barycentric coordinates* and are discussed in these notes.

For a postscript version of these notes look here.

What are Barycentric Coordinates?

Consider a set of points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$ and consider the set of all affine combinations taken from these points. That is all points $\mathbf{P}$ that can be written as

$$\alpha_0 \mathbf{P}_0 + \alpha_1 \mathbf{P}_1 + \cdots + \alpha_n \mathbf{P}_n +$$

for some

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$$

This set of points forms an affine space, and the coordinates

$$(\alpha_0, \alpha_1, \ldots, \alpha_n)$$

are called the barycentric coordinates of the points of the space.

These coordinates system are frequently quite useful, and the interested student will notice that they are used extensively in working with triangles. In many cases (e.g. on a line, as shown below), this barycentric parameterization is exactly the parameterization that we usually use.

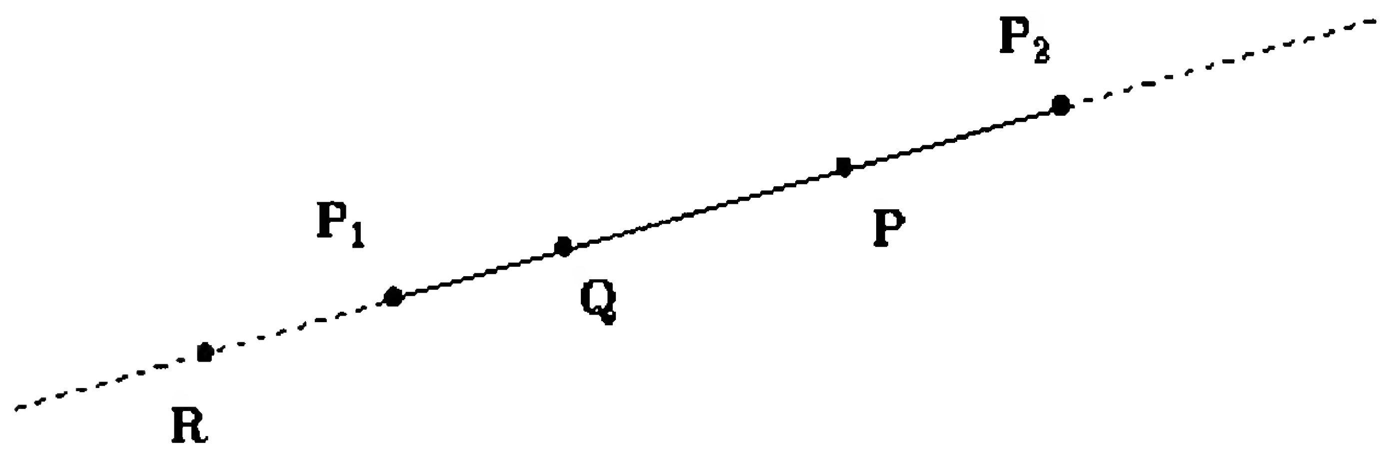
Example -- Point on a Line Segment

To give a simple example of barycentric coordinates, consider two points $\mathbf{P}_1$ and $\mathbf{P}_2$ in the plane. If α_1 and α_2 are scalars such that $\alpha_1 + \alpha_2 = 1$, then the point $\mathbf{P}$ defined by

$$\mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2$$

is a point on the line that passes through $\mathbf{P}_1$ and $\mathbf{P}_2$. If $0 \leq \alpha_1, \alpha_2 \leq 1$ then the point $\mathbf{P}$ is on the line segment joining $\mathbf{P}_1$ and $\mathbf{P}_2$. The following figure shows an example of a line and three points $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$. These points were generated using the following αs:

- $\mathbf{P} : \alpha_1 = \frac{1}{3}, \alpha_2 = \frac{2}{3}$
- $\mathbf{Q} : \alpha_1 = \frac{3}{4}, \alpha_2 = \frac{1}{4}$
- $\mathbf{R} : \alpha_1 = \frac{4}{3}, \alpha_2 = -\frac{1}{3}$



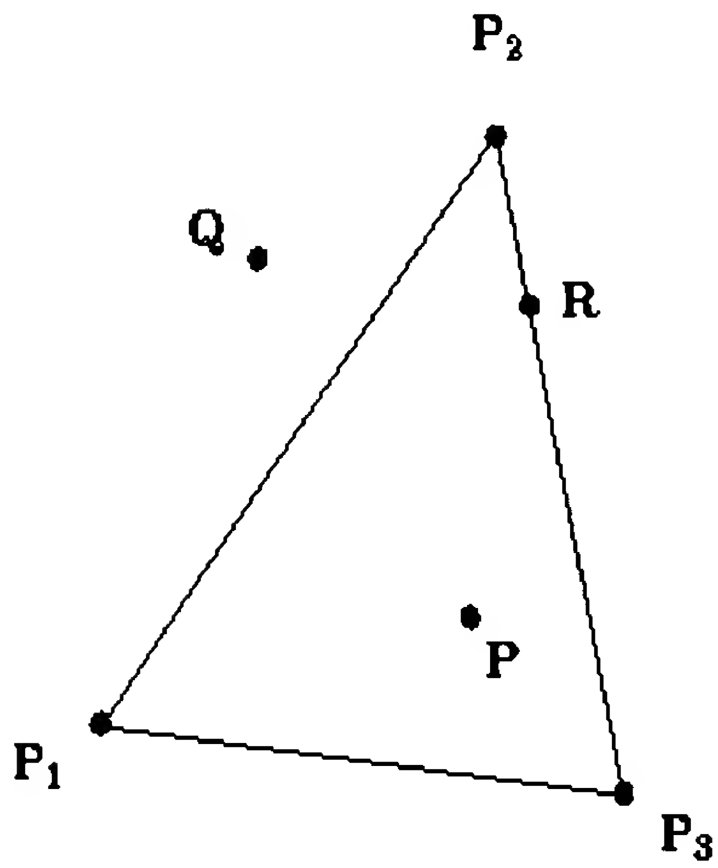
Example -- Point in a Triangle

To give a slightly more complex example of barycentric coordinates, consider three points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ in the plane. If α_1, α_2, α_3 are scalars such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$, then the point $\mathbf{P}$ defined by

$$\mathbf{P} = \alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2 + \alpha_3\mathbf{P}_3$$

is a point on the plane of the triangle formed by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$. The point is within the triangle $\triangle\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$ if $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$. If any of the α's is less than zero or greater than one, the point $\mathbf{P}$ is outside the triangle. If any of the α's is zero, we reduce to the example above and note that $\mathbf{P}$ is on one of the lines joining the vertices of the triangle. The following figure shows an example of such a triangle and three points $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$, these points were calculated using the following α's:

- $\mathbf{P} : \alpha_1 = \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{1}{2}.$
- $\mathbf{Q} : \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{4}, \alpha_3 = -\frac{1}{4}.$
- $\mathbf{R} : \alpha_1 = 0, \alpha_2 = \frac{3}{4}, \alpha_3 = \frac{1}{4}.$



Frames and Barycentric Coordinates

There is a natural way to convert the local coordinates of a frame to barycentric coordinates for a certain set of points. Suppose we are given a frame $\mathcal{F} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \mathbf{O})$ for an affine space $\mathcal{A}$. Then we can write any point $\mathbf{P}$ in the space uniquely as

$$\mathbf{P} \ = \ p_1\vec{v}_1 + p_2\vec{v}_2 + \cdots + p_n\vec{v}_n + \mathbf{O}$$

where $(p_1, p_2, ..., p_n)$ are the local coordinates of the point $\mathbf{P}$ with respect to the frame $\mathcal{F}$. If we define the points $\mathbf{P}_i$ by

$$\mathbf{P}_0 = \mathbf{O}$$

$$\mathbf{P}_1 = \mathbf{O} + \vec{v}_1$$

$$\mathbf{P}_2 = \mathbf{O} + \vec{v}_2$$

$$\vdots$$

$$\mathbf{P}_n = \mathbf{O} + \vec{v}_n$$

(i.e., the origin of the frame and the points obtained by adding the coordinate vectors to the origin) and define p_0 to be

$$p_0 \ = \ 1 - (p_1 + p_2 + \cdots + p_n)$$

then we can see that $\mathbf{P}$ can be written as

$$\mathbf{P} = \mathbf{P}_0 + p_1(\mathbf{P}_1 - \mathbf{P}_0) + p_2(\mathbf{P}_2 - \mathbf{P}_0) + \cdots + p_n(\mathbf{P}_n - \mathbf{P}_0)$$

or equivalently, in an affine way as,

$$\mathbf{P} = p_0\mathbf{P}_0 + p_1\mathbf{P}_1 + p_2\mathbf{P}_2 + \cdots + p_n\mathbf{P}_n$$

where $p_0 + p_1 + p_2 + \cdots + p_n = 1$

In this form, the values $(p_0, p_1, p_2, ..., p_n)$ are *barycentric coordinates* of $\mathbf{P}$ relative to the points $(\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n)$

How can Vectors be Represented?

Following the above methods, we can also express the vectors of an affine space in terms of the points. In this case, if we are given the frame $\mathcal{F} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \mathbf{O})$ then for any vector $\vec{u}$, we can write $\vec{u}$ as

$$\vec{u} = u_1\vec{v}_1 + u_2\vec{v}_2 + \cdots + u_n\vec{v}_n$$

for some constants $u_1, u_2, ..., u_n$ (since the vectors of the frame are assumed to be linear independent). Now, if we define

$$u_0 = -(u_1 + u_2 + \cdots + u_n)$$

If we define the points $\mathbf{P}_i$ by

$$\mathbf{P}_0 = \mathbf{O}$$

$$\mathbf{P}_1 = \mathbf{O} + \vec{v}_1$$

$$\mathbf{P}_2 = \mathbf{O} + \vec{v}_2$$

$$\vdots$$

$$\mathbf{P}_n = \mathbf{O} + \vec{v}_n$$

then

$$\vec{u} = u_1(\mathbf{P}_1 - \mathbf{P}_0) + u_2(\mathbf{P}_2 - \mathbf{P}_0) + \cdots + u_n(\mathbf{P}_n - \mathbf{P}_0)$$

or equivalently, in an affine way as,

$$\vec{u} = u_0\mathbf{P}_0 + u_1\mathbf{P}_1 + u_2\mathbf{P}_2 + \cdots + u_n\mathbf{P}_n$$

where now we have that $u_0 + u_1 + u_2 + \cdots + u_n = 0$.

No References!

Summary

Barycentric coordinates are another important method of introducing coordinates into an affine space. If the coordinates sum to one, they represent a point ; if the coordinates sum to zero, they represent a vector.

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Graphics Notes

UC Davis Graphics Group

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On-Line Computer Graphics Notes

Affine Combinations

Overview

An affine space contains both points and vectors. Points are typically used to position ourselves in space and vectors are use to move about in space. However in the definition of an affine space, the operations on vectors are numerous, while the operations on points are sparse.

Here we define a fundamental operation on the points of an affine space, the *affine combination*. This operation is unique on the set of points, as it will be defined just by the points themselves.

For a postscript version of these notes look here.

What is an Affine Combination?

Given a set of points $\mathbf{P}_0, \mathbf{P}_1, ..., \mathbf{P}_n$ in an affine space, an affine combination is defined to be the point

$$\alpha_0 \mathbf{P}_0 + \alpha_1 \mathbf{P}_1 + \cdots + \alpha_n \mathbf{P}_n$$

where the α_i are scalars and

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$$

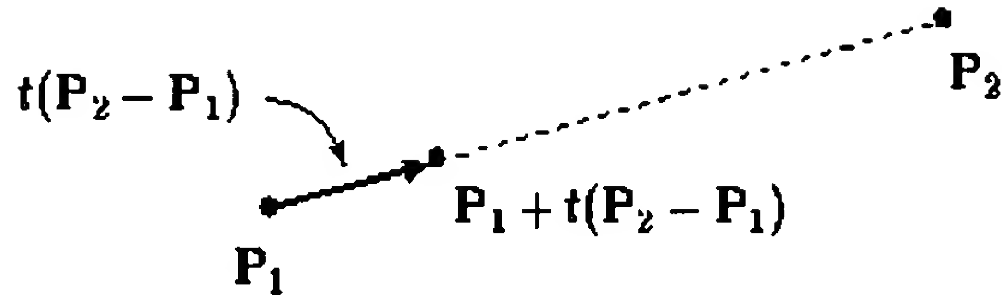
We note that an affine combination of points defines a new point in space. It is *not* a general linear combination, but a linear combination of the points where the scalars must sum to one.

An Affine Combination of Two Points

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be points in an affine space. Consider the expression

$$\mathbf{P} = \mathbf{P}_1 + t(\mathbf{P}_2 - \mathbf{P}_1)$$

By the axioms of an affine space this equation is meaningful, as $\mathbf{P}_2 - \mathbf{P}_1$ is a vector, and therefore so is $t(\mathbf{P}_2 - \mathbf{P}_1)$. Therefore $\mathbf{P}$ is the sum of a point and a vector, which is again a point in the affine space This point $\mathbf{P}$ represents a point on the ``line'' that passes through $\mathbf{P}_1$ and $\mathbf{P}_2$.



We note that if $0 \leq t \leq 1$ then $\mathbf{P}$ is somewhere on the ``line segment'' joining $\mathbf{P}_1$ and $\mathbf{P}_2$.

This expression allows us to define the affine combination on two points. We define

$$(1-t)\mathbf{P}_1 + t\mathbf{P}_2$$

to be the point defined by

$$\mathbf{P} \;=\; \mathbf{P}_1 + t(\mathbf{P}_2 - \mathbf{P}_1)$$

We can then define an **affine combination** $\mathbf{P}$ of two points $\mathbf{P}_1$ and $\mathbf{P}_2$ to be

$$\mathbf{P} \;=\; \alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2$$

where $\alpha_1 + \alpha_2 = 1$.

The form $\mathbf{P} = (1-t)\mathbf{P}_1 + t\mathbf{P}_2$ is shown to be an affine transformation by setting $\alpha_2 = t$. The form $\mathbf{P} = \alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2$ can be seen to be equivalent to the form $\mathbf{P} = \mathbf{P}_1 + t(\mathbf{P}_2 - \mathbf{P}_1)$, by setting $t = \frac{\alpha_2}{\alpha_1 + \alpha_2}$.

An Affine Combination of an Arbitrary Number of Points

We can use the above argument this to define an affine combination of an arbitrary number of points. If $\mathbf{P}_1, \mathbf{P}_2, ..., \mathbf{P}_n$ are points and $\alpha_1, \alpha_2, ..., \alpha_n$ are scalars such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then

$$\alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2 + \cdots + \alpha_n\mathbf{P}_n$$

is defined to be equivalent to the point

$$\mathbf{P}_1 + \alpha_2(\mathbf{P}_2 - \mathbf{P}_1) + \cdots + \alpha_n(\mathbf{P}_n - \mathbf{P}_1)$$

which uses the fact that $\alpha_1 = 1 - \alpha_2 - \cdots - \alpha_n$.

Example -- Affine Combinations in Triangles

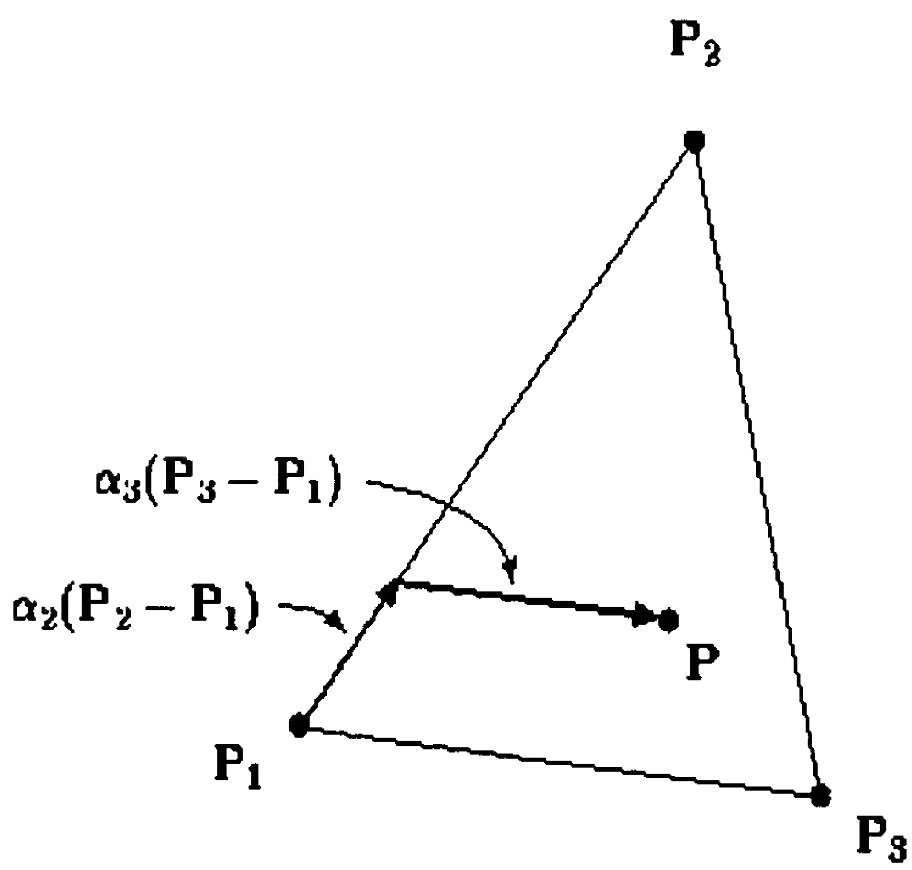
To construct an excellent example of an affine combination consider three points $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$. A point $\mathbf{P}$ defined by

$$\mathbf{P} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \alpha_3 \mathbf{P}_3$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$, gives a point in the triangle $\Delta \mathbf{P}_1 \mathbf{P}_2 \mathbf{P}_3$. We note that the definition of affine combination defines this point to be

$$\mathbf{P} = \mathbf{P}_1 + \alpha_2(\mathbf{P}_2 - \mathbf{P}_1) + \alpha_3(\mathbf{P}_3 - \mathbf{P}_1)$$

The following illustration shows the point $\mathbf{P}$ generated when $\alpha_1 = \alpha_2 = \frac{1}{4}$ and $\alpha_3 = \frac{1}{2}$.



In fact, it can be easily shown that if $0 \leq \alpha 1, \alpha 2, \alpha 3 \leq 1$ then the point $\mathbf{P}$ will be within (or on the boundary) of the triangle. If any α_i is less than zero or greater than one, then the point will lie outside the triangle. If any α_i is zero, then the point will lie on the boundary of the triangle.

No References!

Summary

Affine combinations define a new point from a set of points in an affine space by constructing linear combinations of points with the restrictions that the coefficients of the linear combinations must sum to one. This is the only operation on the points of an affine space. We use this concept extensively in geometric modeling, especially in the definitions of convex combinations and barycentric coordinates.

One should remember that an affine combination is a linear combination but not the other way around.

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